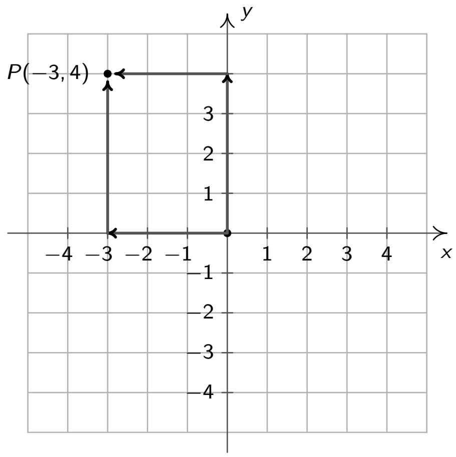
MATH 1700: SECTION 13.1: INTRODUCTION TO POLAR COORDINATES

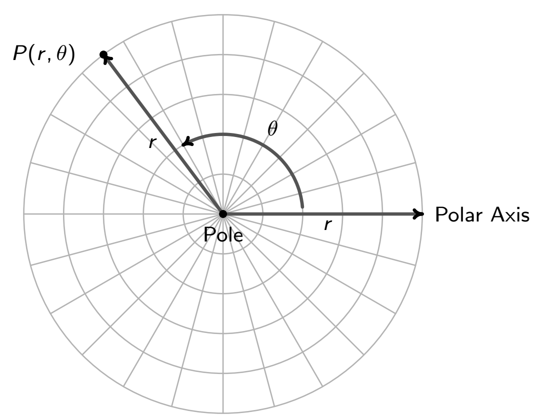
Up to this point, we've been using the Cartesian coordinate plane as a tool to visualize functions. Recall the Cartesian coordinate plane is defined using two number lines – one horizontal and one vertical – which intersect at right angles at a point called the 'origin'.

As seen below on the left, to plot a point with Cartesian coordinates, say $P(-3,4)$, we start at the origin, travel horizontally to the left 3 units, then up 4 units. Alternatively, we could start at the origin, travel up 4 units, then to the left 3 units and arrive at the same location.

For the most part, the 'motions' of the Cartesian system (over and up) describe a rectangle, and most points can be thought of as the corner diagonally across the rectangle from the origin. For this reason, the Cartesian coordinates of a point are often called 'rectangular' coordinates.

In this section, we introduce **polar coordinates** as diagrammed below on the right. We start with an origin point, called the **pole**, and a ray called the **polar axis**.

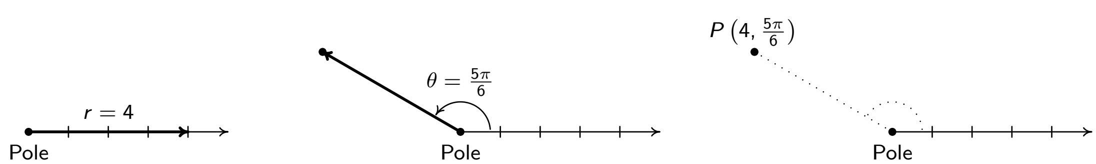




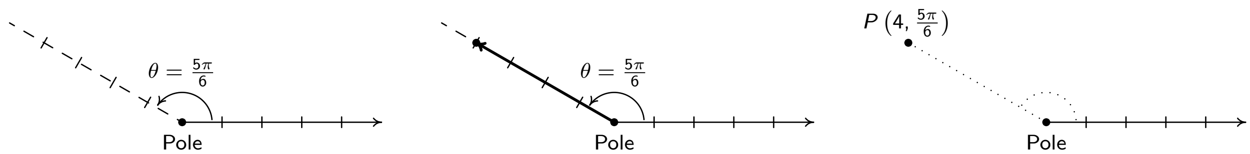
We locate a point P using two coordinates, (r,θ), where r represents a *directed* distance from the pole and θ is a measure of counter-clockwise rotation from the polar axis.

Roughly speaking, the polar coordinates (r,θ) of a point measure 'how far out' the point is from the pole (that's r), and 'how far to rotate' from the polar axis, (that's θ).

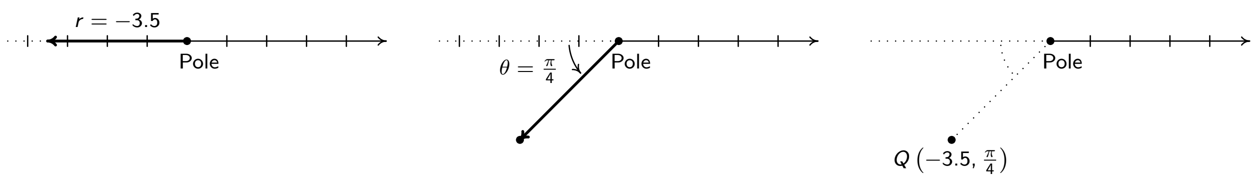
For example, if we wished to plot the point P with polar coordinates $\left(4, \frac{5\pi}{6}\right)$, we'd start at the pole, move out along the polar axis 4 units, then rotate $\frac{5\pi}{6}$ radians counter-clockwise.



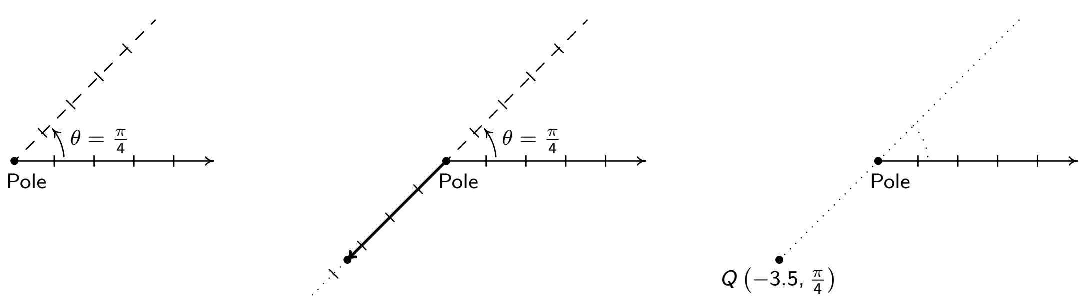
We may also visualize this process by thinking of the rotation first. To plot $P\left(4, \frac{5\pi}{6}\right)$ this way, we rotate $\frac{5\pi}{6}$ counter-clockwise from the polar axis, then move outwards from the pole 4 units. Essentially we are locating a point on the terminal side of $\frac{5\pi}{6}$ which is 4 units away from the pole.



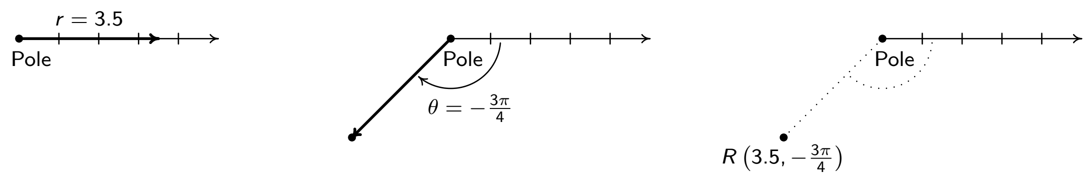
If $r < 0$, we begin by moving in the opposite direction on the polar axis from the pole. For example, to plot the point with polar coordinates $Q\left(-3.5, \frac{\pi}{4}\right)$ we have



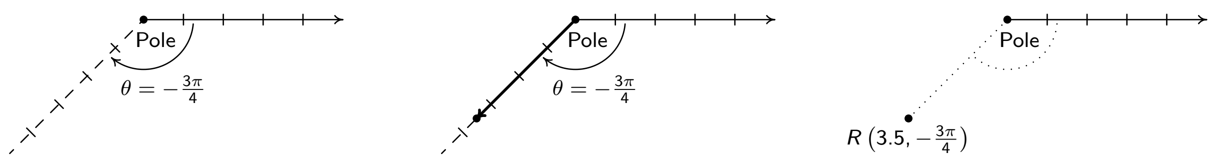
If we interpret the angle first, we rotate $\frac{\pi}{4}$ radians, then move back through the pole 3.5 units. Here we are locating a point 3.5 units away from the pole on the terminal side of $\frac{5\pi}{4}$, not $\frac{\pi}{4}$.



As you may have guessed, $\theta < 0$ means the rotation away from the polar axis is clockwise instead of counter-clockwise. Hence, to plot $R\left(3.5, -\frac{3\pi}{4}\right)$ we have the following.



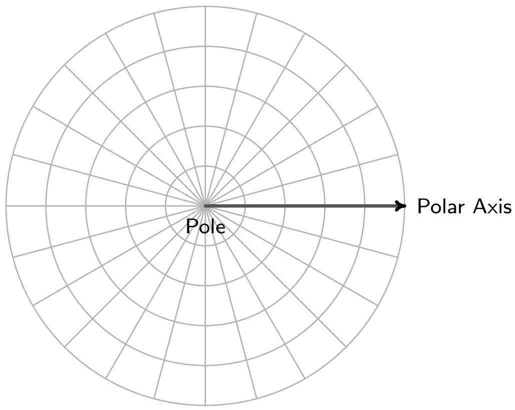
From an 'angles first' approach, we rotate $-\frac{3\pi}{4}$ then move out 3.5 units from the pole. We see that R is the point on the terminal side of $\theta = -\frac{3\pi}{4}$ which is 3.5 units from the pole.



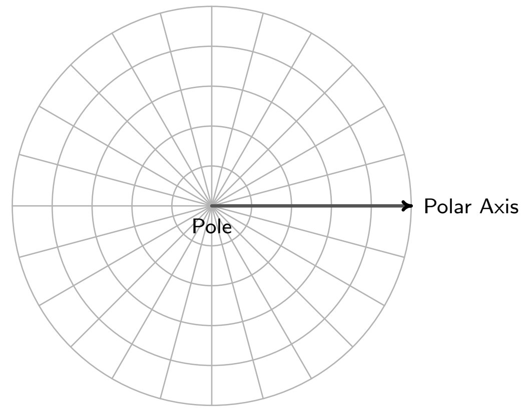
The points Q and R above are, in fact, the same point despite the fact that their polar coordinate representations are different. Unlike Cartesian coordinates where (a, b) and (c, d) represent the same point if and only if $a = c$ and $b = d$, a point can be represented by infinitely many polar coordinate pairs.

EXAMPLE 1: For each point in polar coordinates given below plot the point and then give two additional expressions for the point, one of which has $r > 0$ and the other with $r < 0$.

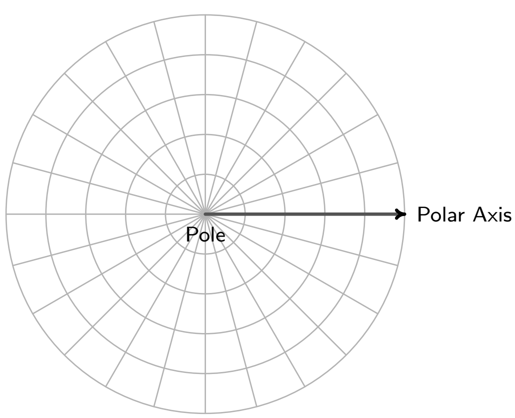
- $P\,(2, 240^\circ)$



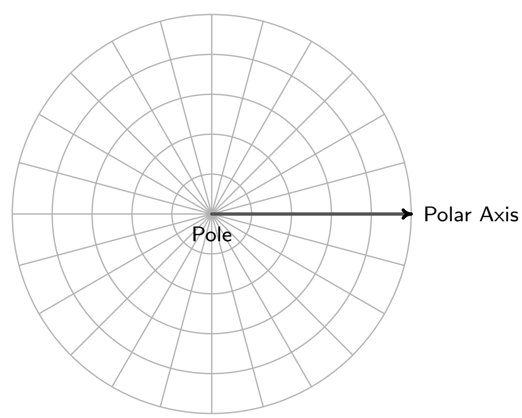
- $P\left(-4, \frac{7\pi}{6}\right)$



- $P\left(117, -\frac{5\pi}{2}\right)$



- $P\left(-3, -\frac{\pi}{4}\right)$



The following result characterizes when two sets of polar coordinates determine the same point in the plane. It could be considered as a definition or a theorem, depending on your point of view. We choose to state it as a property of the polar coordinate system.

EQUIVALENT REPRESENTATIONS OF POINTS IN POLAR COORDINATES:

Suppose (r, θ) and (r', θ') are polar coordinates where $r \neq 0$, $r' \neq 0$ and the angles are measured in radians. Then (r, θ) and (r', θ') determine the same point P if and only if one of the following is true:

- $r' = r$ and $\theta' = \theta + 2\pi k$ for some integer k
- $r' = -r$ and $\theta' = \theta + (2k + 1)\pi$ for some integer k

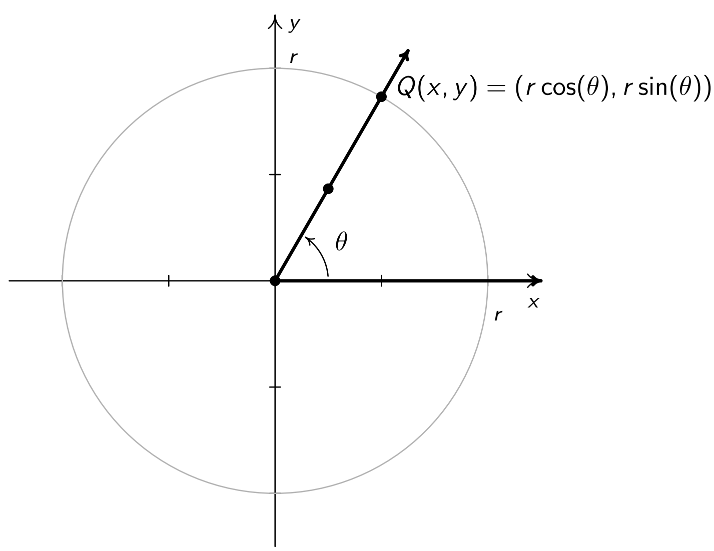
All polar coordinates of the form $(0, \theta)$ represent the pole regardless of the value of θ.

The key to understanding this result is to keep in mind that (r, θ) means (directed distance from pole, angle of rotation). The details of the proof are in the text.

CONVERTING BETWEEN RECTANGULAR AND POLAR COORDINATES:

Suppose P is represented in rectangular coordinates as (x, y) and in polar coordinates as (r, θ). Then

- $x = r\cos(\theta)$ and $y = r\sin(\theta)$
- $x^2 + y^2 = r^2$ and $\tan(\theta) = \dfrac{y}{x}$ (provided $x \neq 0$)



EXAMPLE 2: Convert each point in rectangular coordinates given below into polar coordinates with $r \geq 0$ and $0 \leq \theta < 2\pi$. Use exact values if possible and round any approximate values to two decimal places. Check your answer by converting them back to rectangular coordinates.

1. $P\left(2, -2\sqrt{3}\right)$

2. $Q(-3, -3)$

3. $R(0, -3)$

4. $S(-3, 4)$

Now that we've had practice converting representations of *points* between the rectangular and polar coordinate systems, we now set about converting *equations* from one system to another.

Just as we've used equations in x and y to represent relations in rectangular coordinates, equations in the variables r and θ represent relations in polar coordinates.

EXAMPLE 3:

1. Convert each equation in rectangular coordinates into an equation in polar coordinates.

   (a) $(x-3)^2+y^2=9$

   (b) $y=-x$

   (c) $y=x^2$

2. Convert each equation in polar coordinates into an equation in rectangular coordinates.

   (a) $r = -3$

   (b) $\theta = \frac{4\pi}{3}$

   (c) $r = 1 - \cos(\theta)$

HOMEWORK: Section 13.1: Page 1099: 1 - 93 every other odd.